Autoduality of the compactified Jacobian

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Abstract. We prove the following *autoduality theorem* for an integral projective curve C in any characteristic. Given an invertible sheaf $\mathcal{L}$ of degree 1, form the corresponding Abel map $A_{\mathcal{L}}: C \to \bar{J}$, which maps C into its compactified Jacobian, and form its pullback map $A_{\mathcal{L}}^{*}: \mathrm{Pic}^0_{\bar{J}} \to J$, which carries the connected component of 0 in the Picard scheme back to the Jacobian. If C has, at worst, points of multiplicity 2, then $A_{\mathcal{L}}^{*}$ is an isomorphism, and forming it commutes with specializing C.

Much of our work is valid, more generally, for a family of curves with, at worst, points of embedding dimension 2. In this case, we use the determinant of cohomology to construct a right inverse to $A_{\mathcal{L}}^{*}$. Then we prove a scheme-theoretic version of the theorem of the cube, generalizing Mumford's, and use it to prove that $A_{\mathcal{L}}^{*}$ is independent of the choice of $\mathcal{L}$. Finally, we prove our autoduality theorem: we use the presentation scheme to achieve an induction on the difference between the arithmetic and geometric genera; here, we use a few special properties of points of multiplicity 2.

1. Introduction

Let C be an integral projective curve, defined over an algebraically closed field of any characteristic, and $\mathcal{L}$ an invertible sheaf of degree 1. Form the (generalized) Jacobian, the connected component of the identity of the Picard scheme, $J := \mathrm{Pic}^0_C$. If C is smooth, then J is an Abelian variety, and the Abel map $A_{\mathcal{L}}: C \to J$ is defined by $P \mapsto \mathcal{L}(-P)$. Also, the corresponding pullback is an isomorphism, $A_{\mathcal{L}}^{*}: \mathrm{Pic}^0_J \xrightarrow{\sim} J$, which is independent of the choice of $\mathcal{L}$; thus J is "autodual," or canonically isomorphic to its own dual Abelian variety Pic^0_J. (See Theorem 3 on p. 156 in [**14**] or Proposition 6.9 on p. 118 in [**16**].) Our main result is the autoduality theorem of (2.1); it asserts that, more generally, if C has, at worst,

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double points (arbitrary points of multiplicity 2), then a similar pullback is an isomorphism, and forming it commutes with specializing C.

Suppose first that C has arbitrary singularities. Recall (see [**2**], [**4**], [**5**]) that J has a natural compactification $\bar{J}$, the (fine) moduli space of torsion-free sheaves of rank 1 and degree 0. Also, the Abel map $A_{\mathcal{L}}: C \to \bar{J}$ is defined by $P \mapsto \mathcal{I}_P \otimes \mathcal{L}$ where $\mathcal{I}_P$ is the ideal of P; it is a closed embedding if C is not of genus 0. Furthermore, the Picard scheme $\mathrm{Pic}_{\bar{J}}$ exists and is a union of quasi-projective, open and closed subschemes — including $\mathrm{Pic}^0_{\bar{J}}$ and $\mathrm{Pic}^{\tau}_{\bar{J}}$, which are the connected component of 0 and the subscheme of points with multiples in $\mathrm{Pic}^0_{\bar{J}}$.

Suppose now that all the singularities of C are *surficial*, that is, of embedding dimension 2. Recall (see [**1**]) that $\bar{J}$ is rather nice; it is a local complete intersection, and is integral and projective. So forming $\mathrm{Pic}_{\bar{J}}$ commutes with specializing C, but conceivably forming $\mathrm{Pic}^0_{\bar{J}}$ does not. Nevertheless, we prove two general results, Propositions (2.2) and (3.7). The former asserts that $A^*_{\mathcal{L}}: \mathrm{Pic}^0_{\bar{J}} \to J$ has a natural right inverse β, which is independent of the choice of $\mathcal{L}$. The latter is much deeper, and asserts that $A^*_{\mathcal{L}}$ is itself independent of the choice of $\mathcal{L}$.

Suppose finally that all the singularities of C are double points. Then $A^*_{\mathcal{L}}$ is an isomorphism and $\mathrm{Pic}^0_{\bar{J}} = \mathrm{Pic}^{\tau}_{\bar{J}}$ by our autoduality theorem. Now, double points are surficial; hence $\bar{J}$ is integral. Moreover, there exists a scheme parameterizing the torsion-free rank-1 sheaves on $\bar{J}$; it is a fine moduli space, and its connected components are projective. Let $\overline{U}$ denote the closure of $\mathrm{Pic}^0_{\bar{J}}$. Then the isomorphism $A^*_{\mathcal{L}}: \mathrm{Pic}^0_{\bar{J}} \xrightarrow{\sim} J$ extends to a map $\eta: \overline{U} \to \bar{J}$ by Corollary (4.4); this is our deepest result, and rests on everything preceding it. Is η an isomorphism? Perhaps yes, perhaps no; our work does not appear to suggest which.

All four of our results are compatible with specializing C. More precisely, we prove relative versions of them for flat, projective families of geometrically integral curves over an arbitrary locally Noetherian base scheme. Some fibers may be smooth, others not. A node may degenerate into a cusp; two nodes may coalesce into a tac.

What happens when all the singularities of C are *surficial*? Is $A^*_{\mathcal{L}}$ an isomorphism then too? The evidence is mixed. On the one hand, Propositions (2.2) and (3.7) suggest so, as they assert that $A^*_{\mathcal{L}}$ has a right inverse β, and both maps are independent of the choice of $\mathcal{L}$. On the other hand, our proof of the autoduality theorem suggests not; it doesn't simply fail when C has singularities of higher multiplicity, rather it suggests that then there may be a counterexample.

Indeed, to prove the autoduality theorem, we proceed basically as follows. We form $J^1 := \mathrm{Pic}^1_C$, the component of the Picard scheme that parameterizes the invertible sheaves of degree 1. Then we put together the Abel maps $A_{\mathcal{L}}$, as $\mathcal{L}$ varies, to form the Abel map of bidegree (1,1):

$$A: C \times J^1 \to \bar{J}.$$

This map is studied in the authors' paper [**9**] (where, however, the two factors are taken in the opposite order; that is, A maps $J^1 \times C$ into $\bar{J}$). In [**9**], the following facts are proved. Suppose that C is Gorenstein. Then A is smooth; so its image V is open. Furthermore, if g denotes the arithmetic genus, then the complement

$\bar{J} - V$ is of dimension at most $g - 2$ if and only if all the singularities of C are double points.

Hence, if C has higher surficial singularities, then $\bar{J}$ is irreducible, and $\bar{J} - V$ contains a set of codimension 1. This set could support a Cartier divisor D. If D exists, then $A_{\mathcal{L}}^{*}\mathcal{O}(D)$ is trivial for any $\mathcal{L}$. Furthermore, D could vary in an algebraic family with support on $\bar{J} - V$ and with two linearly inequivalent members. If so, then their difference would correspond to a point of $\mathrm{Pic}^{0}_{\bar{J}}$, other than 0. Thus $A_{\mathcal{L}}^{*}$ would not be injective, and we'd have a counterexample.

On the other hand, suppose that all the singularities of C are double points. Then $\bar{J} - V$ is small. Moreover, $\bar{J}$ is a local complete intersection. So we may (and will) prove that $A_{\mathcal{L}}^{*}$ is injective basically as follows. Let $\mathcal{N}$ be an invertible sheaf on $\bar{J}$. Then $\mathcal{N}$ is trivial if its restriction $\mathcal{N}|V$ is trivial. In turn, to show that $\mathcal{N}|V$ is trivial, we may use descent theory since A is smooth, so flat.

Suppose that $\mathcal{N}$ corresponds to a point of $\mathrm{Pic}^{0}_{\bar{J}}$. Then there are invertible sheaves $\mathcal{N}_1$ on C and $\mathcal{N}_2$ on J^1 such that

$$A^{*}\mathcal{N} = p_1^{*}\mathcal{N}_1 \otimes p_2^{*}\mathcal{N}_2,$$

where the p_i are the projections; the existence of the $\mathcal{N}_i$ results from our general theory of the theorem of the cube, especially Part (2) of Lemma (3.6). Consequently, $A_{\mathcal{L}}^{*}\mathcal{N}$ is equal to $\mathcal{N}_1$, so is independent of the choice of $\mathcal{L}$, as was asserted above.

Suppose also that $A_{\mathcal{L}}^{*}\mathcal{N}$ is trivial for some $\mathcal{L}$. We have to prove that $\mathcal{N}$ is trivial too. To begin, note that $\mathcal{N}_1$ is trivial, so $A^{*}\mathcal{N}$ is equal to $p_2^{*}\mathcal{N}_2$. We proceed by induction on the arithmetic genus g. Choose a double point Q on C, and blow Q up, getting $\varphi\colon C^{\dagger} \to C$. Then $C^{\dagger}$ too has only double points by Lemma (6.4) of [**9**], and its arithmetic genus is $g - 1$ by Proposition (6.1) of [**9**].

To relate the compactified Jacobians $\bar{J}_C$ and $\bar{J}_{C^{\dagger}}$ of C and $C^{\dagger}$, we use the presentation scheme P and the maps $\kappa\colon P \to \bar{J}_C$ and $\pi\colon P \to \bar{J}_{C^{\dagger}}$; they are studied in [**9**]. By Theorem (6.3) of [**9**], because Q is a double point, π is a locally trivial $\mathbf{P}^1$-bundle. On each $\mathbf{P}^1$, the restriction of $\kappa^{*}\mathcal{N}$ is trivial because $\mathcal{N}$ corresponds to a point of $\mathrm{Pic}^{0}_{\bar{J}_C}$. Hence, $\kappa^{*}\mathcal{N}$ is the pullback of a sheaf $\mathcal{N}^{\dagger}$ on $\bar{J}_{C^{\dagger}}$.

Because C and $C^{\dagger}$ are Gorenstein, there are two natural commutative diagrams

$$\begin{array}{ccc} C^{\dagger} \times J^1_C & \xrightarrow{\Lambda} & P \\ {\scriptstyle 1\times\varphi^{*}}\downarrow & & \downarrow{\scriptstyle \pi} \\ C^{\dagger} \times J^1_{C^{\dagger}} & \xrightarrow{A_{C^{\dagger}}} & \bar{J}_{C^{\dagger}} \end{array} \qquad \begin{array}{ccc} C^{\dagger} \times J^1_C & \xrightarrow{\Lambda} & P \\ {\scriptstyle \varphi\times 1}\downarrow & & \downarrow{\scriptstyle \kappa} \\ C \times J^1_C & \xrightarrow{A_C} & \bar{J}_C \end{array}$$

by Corollary (5.5) in [**9**]; here A_C and $A_{C^{\dagger}}$ are the Abel maps of C and $C^{\dagger}$. The diagrams imply that, if $\mathcal{L}^{\dagger} := \varphi^{*}\mathcal{L}$, then $A_{\mathcal{L}^{\dagger}}^{*}\mathcal{N}^{\dagger}$ is equal to $\varphi^{*}A_{\mathcal{L}}^{*}\mathcal{N}$, which is trivial by hypothesis.

Since autoduality holds for $C^{\dagger}$ by induction, $\mathcal{N}^{\dagger}$ is trivial. Since the pullback of $\mathcal{N}^{\dagger}$ to P is equal to $\kappa^{*}\mathcal{N}$, the latter is trivial. So thanks to the commutativity of the second diagram above, $(\varphi \times 1)^{*}A_C^{*}\mathcal{N}$ is trivial. Since $A_C^{*}\mathcal{N}$ is equal to $p_2^{*}\mathcal{N}_2$, it follows that $\mathcal{N}_2$ is trivial.

Since C and $C^{\dagger}$ are Gorenstein, by Corollary (5.5) in [**9**], the second diagram above is Cartesian. Consider the descent data on $(\varphi \times 1)^{*}A_C^{*}\mathcal{N}$ with respect to Λ;

since $\kappa^*\mathcal{N}$ is trivial, this data is trivial. Hence, so is that on $A_C^*\mathcal{N}$ with respect to A_C because κ is birational. Therefore $\mathcal{N}$ is trivial. Thus $A_{\mathcal{L}}^*$ is injective.

We construct the right inverse $\beta: J \to \mathrm{Pic}^0_{\bar{J}}$ to $A_{\mathcal{L}}^*$ by using the determinant of cohomology $\mathcal{D}$ along the projection $q_2: C \times \bar{J} \to \bar{J}$. We proceed as follows. Fix a universal sheaf $\mathcal{I}$ on $C \times \bar{J}$. Then, given any invertible sheaf $\mathcal{M}$ on C of degree 0, set

$$\beta(\mathcal{M}) := (\mathcal{D}(\mathcal{I} \otimes q_1^*\mathcal{M}))^{-1} \otimes \mathcal{D}(\mathcal{I}),$$

where $q_1: C \times \bar{J} \to C$ is the projection.

This construction was suggested by Breen [pvt. comm., 1985]. It is a modern formulation of an older construction using the theta divisor. Namely,

$$\beta(\mathcal{M}) = \Theta_{\mathcal{L}} - \tau_{\mathcal{M}}^*\Theta_{\mathcal{L}}$$

where $\tau_{\mathcal{M}}: \bar{J} \to \bar{J}$ is the translation, given by tensoring with $\mathcal{M}$, and where $\Theta_{\mathcal{L}}$ is the divisor obtained by pulling back the canonical theta divisor along the isomorphism $\bar{J} \xrightarrow{\sim} \bar{J}^{g-1}$ given by tensoring with $\mathcal{L}^{g-1}$. We consider the equivalence of the two formulations in more detail in Remark (2.4).

Since $A_{\mathcal{L}}^*\beta = 1$, the map β is a closed embedding. Since $A_{\mathcal{L}}^*$ is injective, we could conclude that it is an isomorphism if we knew, a priori, that $\mathrm{Pic}^0_{\bar{J}}$ is reduced. We don't. So we must prove that $A_{\mathcal{L}}^*$ is a monomorphism, that is, injective on T-points; we take care to do so in (4.1).

In short, in Section 2, we formulate the autoduality theorem, our main result: if the curves in a family have double points at worst, then the Abel map $A_{\mathcal{L}}^*$ is an isomorphism. Then we treat β, which is the canonical right inverse to $A_{\mathcal{L}}^*$. In Section 3, we generalize Mumford's scheme-theoretic theorem of the cube, and conclude that $A_{\mathcal{L}}^*$ is independent of the choice of $\mathcal{L}$. Finally, in Section 4, we prove our autoduality theorem, and then extend $A_{\mathcal{L}}^*$ to a map from the natural compactification of $\mathrm{Pic}^0_{\bar{J}}$ onto $\bar{J}$.

2. Autoduality

(2.1) *Statement.* Consider a *flat projective family of integral curves* $p: C \to S$; that is, S is a locally Noetherian scheme, and p is a flat and projective map with geometrically integral fibers of dimension 1. Recall (see [**2**], [**4**], [**5**]) that, given an integer n, there exists a projective S-scheme $\bar{J}^n_{C/S}$ that parameterizes the torsion-free rank-1 sheaves of degree n on the fibers of C/S. Furthermore, there exists an open subscheme $J^n_{C/S}$ parameterizing those sheaves that are invertible. Also, forming $\bar{J}^n_{C/S}$ and $J^n_{C/S}$ commutes with changing the base S. As is customary, call $J^n_{C/S}$ the (relative generalized) *Jacobian* of C/S, and $\bar{J}^n_{C/S}$ the *compactified Jacobian*. We will often abbreviate $J^n_{C/S}$ by J^n and $\bar{J}^n_{C/S}$ by $\bar{J}^n$. Set

$$J_{C/S} := J^0_{C/S} \text{ and } \bar{J}_{C/S} := \bar{J}^0_{C/S}.$$

We will also abbreviate $J_{C/S}$ by J and $\bar{J}_{C/S}$ by $\bar{J}$.

More precisely, a (relative) *torsion-free rank-1 sheaf* $\mathcal{I}$ on C/S is an S-flat coherent $\mathcal{O}_C$-module $\mathcal{I}$ such that, for each point s of S, the fiber $\mathcal{I}(s)$ is a torsion-free

rank-1 sheaf on the fiber $C(s)$. Moreover, $\mathcal{I}$ is *of degree* n if $\mathcal{I}(s)$ satisfies the relation,

$$\chi(\mathcal{I}(s)) - \chi(\mathcal{O}_{C(s)}) = n.$$

Given a locally Noetherian S-scheme T, a torsion-free rank-1 sheaf of degree n on $C \times T/T$ defines an S-map $T \to \bar{J}^n$. Conversely, every such S-map arises from such a sheaf, which is determined up to tensor product with the pullback of an invertible sheaf on T, at least if the smooth locus of C/S admits a section. If so, then in particular the identity map $1_{\bar{J}^n}$ arises from such a sheaf on $C \times \bar{J}^n/\bar{J}^n$; the latter sheaf is known as a *universal* (or Poincaré) sheaf, as any $T \to \bar{J}^n$ arises from the sheaf on $C \times T/T$ obtained by pulling back a universal sheaf.

In general, an S-map $T \to \bar{J}^n$ arises rather from a pair $(T'/T, \mathcal{I}')$ where T'/T is an étale covering (that is, the map $T' \to T$ is étale, surjective, and of finite type) and where $\mathcal{I}'$ is a torsion-free rank-1 sheaf of degree n on $C \times T'/T'$. Such a pair defines such an S-map if and only if there is an étale covering $T'' \big/ T' \times_T T'$ such that the two pullbacks of $\mathcal{I}'$ to $C \times T''$ are equal. A second such pair $(T_1'/T, \mathcal{I}_1')$ defines the same S-map if and only if there is an étale covering $T'' \big/ T' \times_T T_1'$ such that the pullbacks of $\mathcal{I}'$ and $\mathcal{I}_1'$ to $C \times T''$ are equal. In sum, $\bar{J}^n$ represents the étale sheaf associated to the functor of torsion-free rank-1 sheaves.

Given an invertible sheaf $\mathcal{L}$ of degree 1 on C/S, define the *Abel map,*

$$A_{\mathcal{L}}: C \to \bar{J},$$

as follows. Let $\mathcal{I}_\Delta$ be the ideal of the diagonal Δ of $C \times C$, and $p_1: C \times C \to C$ be the first projection. Then $\mathcal{I}_\Delta$ is a torsion-free rank-1 sheaf of degree -1 on $C \times C/C$, and the tensor product $\mathcal{I}_\Delta \otimes p_1^*\mathcal{L}$ defines $A_{\mathcal{L}}$. Forming $A_{\mathcal{L}}$ commutes with changing the base S, and if the fibers of C/S are not of arithmetic genus 0, then $A_{\mathcal{L}}$ is a closed embedding by [**5**, (8.8), p. 108].

Assume now that the geometric fibers of C/S have only surficial singularities (ones with embedding dimension 2), for example, double points. Then the projective S-scheme $\bar{J}^n$ is flat, and its geometric fibers are integral local complete intersections; see [**1**, (9), p. 8]. Hence, the Picard scheme $\mathrm{Pic}_{\bar{J}^n/S}$ exists and is a disjoint union of quasi-projective S-schemes; see Théorème 3.1, p. 232-06, in [**10**], and Corollary (6.7)(ii), p. 96, in [**5**]. So the Abel map induces an S-map,

$$A_{\mathcal{L}}^*: \mathrm{Pic}_{\bar{J}/S} \to \textstyle\coprod_n J^n.$$

As is customary [**10**, p. 236-03], let $\mathrm{Pic}^0_{\bar{J}/S}$ denote the set-theoretic union of the connected components of the identity 0 in the fibers of $\mathrm{Pic}_{\bar{J}/S}$, and let $\mathrm{Pic}^\tau_{\bar{J}/S}$ denote the set of points of $\mathrm{Pic}_{\bar{J}/S}$ that have a multiple in $\mathrm{Pic}^0_{\bar{J}/S}$. The set $\mathrm{Pic}^\tau_{\bar{J}/S}$ is open; give it the induced scheme structure.

The following theorem asserts that, if the geometric fibers of C/S only have double points (of arbitrary order) as singularities, then $\mathrm{Pic}^0_{\bar{J}/S}$ and $\mathrm{Pic}^\tau_{\bar{J}/S}$ are equal, and under $A_{\mathcal{L}}^*$, they are isomorphic to J. This is our main result, and its proof occupies the rest of the paper.

Theorem (Autoduality). *Let C/S be a flat projective family of integral curves. Assume its geometric fibers have double points at worst. Then* $\mathrm{Pic}^0_{\bar{J}/S} = \mathrm{Pic}^\tau_{\bar{J}/S}$.

Furthermore, the Abel map induces an isomorphism,

$$A^*_{\mathcal{L}}\colon \operatorname{Pic}^\tau_{\bar{J}/S} \xrightarrow{\sim} J,$$

which is independent of the choice of the invertible sheaf $\mathcal{L}$ of degree 1 on C/S; in fact, the isomorphism exists whether or not any sheaf $\mathcal{L}$ does.

Proposition (2.2) (Right inverse). *Let C/S be a flat projective family of integral curves. Assume its geometric fibers only have surficial singularities. Then there exists a natural map,*

$$\beta\colon J \to \operatorname{Pic}_{\bar{J}/S},$$

whose formation commutes with base change, and whose image lies in the subset $\operatorname{Pic}^0_{\bar{J}/S}$. Furthermore, $A^*_{\mathcal{L}} \circ \beta = 1_J$ for any $\mathcal{L}$.

Proof. Choose an étale covering S'/S such that the smooth locus of $C \times S'/S'$ admits a section (such a covering exists by [**11**, IV$_4$ 17.16.3(ii), p. 106]). Choose universal sheaves $\mathcal{I}$ on $C \times \bar{J} \times S'$ and $\mathcal{M}$ on $C \times J \times S'$. Form $C \times \bar{J} \times J \times S'$, and let p_{ijk} be the projection onto the product of the indicated factors. Set

$$\mathcal{M}^\diamond := (\mathcal{D}_{p_{234}}(p^*_{124}\mathcal{I} \otimes p^*_{134}\mathcal{M}))^{-1} \otimes \mathcal{D}_{p_{234}}(p^*_{124}\mathcal{I}) \text{ on } \bar{J} \times J \times S'$$

where $\mathcal{D}_{p_{234}}$ denotes the determinant of cohomology; see Section 6 in [**8**], or [**13**]. So $\mathcal{M}^\diamond$ is an invertible sheaf. It defines the desired map β as we now prove.

The sheaf $\mathcal{I}$ is determined up to tensor product with the pullback of an invertible sheaf $\mathcal{N}$ on $\bar{J} \times S'$. So the projection formula for the determinant of cohomology yields

$$\mathcal{D}_{p_{234}}(p^*_{124}\mathcal{I} \otimes p^*_{24}\mathcal{N} \otimes p^*_{134}\mathcal{M}) = \mathcal{D}_{p_{234}}(p^*_{124}\mathcal{I} \otimes p^*_{134}\mathcal{M}) \otimes p^*_{13}\mathcal{N}^{\otimes m}$$
$$\mathcal{D}_{p_{234}}(p^*_{124}\mathcal{I} \otimes p^*_{24}\mathcal{N}) = \mathcal{D}_{p_{234}}(p^*_{124}\mathcal{I}) \otimes p^*_{13}\mathcal{N}^{\otimes n}$$

where the p's are the indicated projections and where m and n are the Euler characteristics of $p^*_{124}\mathcal{I} \otimes p^*_{134}\mathcal{M}$ and $p^*_{124}\mathcal{I}$ on the fibers of p_{234} (thus m and n are locally constant functions on $\bar{J} \times J \times S'$). Now, $m = n$ because the fibers of $p^*_{134}\mathcal{M}$ have degree 0. Therefore $\mathcal{M}^\diamond$ does not depend on the choice of $\mathcal{I}$.

Similarly, the sheaf $\mathcal{M}$ is determined up to tensor product with the pullback of an invertible sheaf $\mathcal{P}$ on $J \times S'$. Moreover, the preceding argument shows that, if $\mathcal{M}$ is replaced by its tensor product with the pullback of $\mathcal{P}$, then $\mathcal{M}^\diamond$ is replaced by its tensor product with the pullback of $\mathcal{P}^{\otimes m}$.

Set $S'' := S' \times S'$. There are two pullbacks of $\mathcal{I}$ to $C \times \bar{J} \times S''$, and both are universal sheaves. Similarly, there are two pullbacks of $\mathcal{M}$ to $C \times J \times S''$, and both are universal sheaves. Now, forming the determinant of cohomology commutes with changing the base. Therefore, by the preceding paragraphs, the two pullbacks of $\mathcal{M}^\diamond$ to $\bar{J} \times J \times S''$ differ by tensor product with the pullback of an invertible sheaf on $J \times S''$. Hence $\mathcal{M}^\diamond$ defines a map $\beta\colon J \to \operatorname{Pic}_{\bar{J}/S}$.

Consider another choice of covering S'_1/S and of sheaves $\mathcal{I}_1$ and $\mathcal{M}_1$, and form the corresponding $\mathcal{M}^\diamond_1$. Set $S'' := S' \times S'_1$. Then the pullbacks of $\mathcal{I}_1$ and $\mathcal{I}$ to $C \times \bar{J} \times S''$ are both universal. Similarly, the pullbacks of $\mathcal{M}_1$ and $\mathcal{M}$ to $C \times J \times S''$ are both universal. Hence, by the preceding argument, the pullbacks of $\mathcal{M}^\diamond$ and

$\mathcal{M}_1^\diamond$ to $\bar{J} \times J \times S''$ differ by tensor product with the pullback of an invertible sheaf on $J \times S''$. So $\mathcal{M}_1^\diamond$ and $\mathcal{M}^\diamond$ define the same map β.

Forming β commutes with changing S since forming the determinant does.

The image of β lies in $\mathrm{Pic}^0_{\bar{J}/S}$. Indeed, we may change the base to an arbitrary geometric point of S, and so work over an algebraically closed field. Then J is integral. So it suffices to prove $\beta(0) = 0$. Now, we may choose $\mathcal{I}$ on $C \times \bar{J}$ and $\mathcal{M}$ on $C \times J$. Then the fiber $\mathcal{M}(0)$ is equal to $\mathcal{O}_C$. Since forming the determinant commutes with passing to the fiber, it follows that $\mathcal{M}^\diamond(0) = \mathcal{O}_{\bar{J}}$. So $\beta(0) = 0$.

Finally, $A_{\mathcal{L}}^* \circ \beta = 1_J$. Indeed, it suffices to check this equation after changing the base to S'; so assume $S' = S$. Then $\mathcal{I}$ sits on $C \times \bar{J}$, and $\mathcal{M}$ sits on $C \times J$. So $A_{\mathcal{L}}$ is defined by $(1_C \times A_{\mathcal{L}})^*\mathcal{I}$, as well as by $\mathcal{I}_\Delta \otimes p_1^*\mathcal{L}$. Hence these two sheaves differ by tensor product with the pullback, along the projection p_2, of an invertible sheaf on C. It follows as above from the properties of the determinant of cohomology that

$$(A_{\mathcal{L}} \times 1_J)^*\mathcal{M}^\diamond = (\mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}_\Delta \otimes p_1^*\mathcal{L} \otimes p_{13}^*\mathcal{M}))^{-1} \otimes \mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}_\Delta \otimes p_1^*\mathcal{L}) \quad (2.2.1)$$

on $C \times J$. So both sides of this equation define the same map $J \to \coprod_n J^n$.

To evaluate the right-hand side of (2.2.1), consider the natural sequence,

$$0 \to \mathcal{I}_\Delta \to \mathcal{O}_{C \times C} \to \mathcal{O}_\Delta \to 0. \quad (2.2.2)$$

Pull it back to $C \times C \times J$, then tensor with $p_1^*\mathcal{L} \otimes p_{13}^*\mathcal{M}$ and with $p_1^*\mathcal{L}$. The additivity of the determinant of cohomology now yields

$$\mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}_\Delta \otimes p_1^*\mathcal{L} \otimes p_{13}^*\mathcal{M}) = \mathcal{D}_{p_{23}}(p_1^*\mathcal{L} \otimes p_{13}^*\mathcal{M}) \otimes (p_1^*\mathcal{L} \otimes \mathcal{M})^{-1},$$
$$\mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}_\Delta \otimes p_1^*\mathcal{L}) = \mathcal{D}_{p_{23}}(p_1^*\mathcal{L}) \otimes (p_1^*\mathcal{L})^{-1}.$$

Consider the following Cartesian square:

$$\begin{tikzcd} C \times J \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & C \times C \times J \arrow[l, "p_{13}"'] \arrow[d, "p_{23}"] \\ J & C \times J \arrow[l, "p_2"'] \end{tikzcd}$$

Forming the determinant commutes with changing the base. So, on $C \times J$,

$$\mathcal{D}_{p_{23}}(p_1^*\mathcal{L} \otimes p_{13}^*\mathcal{M}) = p_2^*\mathcal{D}_{p_2}(p_1^*\mathcal{L} \otimes \mathcal{M}),$$
$$\mathcal{D}_{p_{23}}(p_1^*\mathcal{L}) = p_2^*\mathcal{D}_{p_2}(p_1^*\mathcal{L}).$$

Hence the right-hand side of (2.2.1) differs from $\mathcal{M}$ by tensor product with the pullback of an invertible sheaf on J. Therefore, $A_{\mathcal{L}}^* \circ \beta = 1_J$, and the proof is complete.

Remark (2.3). In Proposition (2.2), we made the hypothesis that the fibers of C/S have surficial singularities, but we did not use the hypothesis directly in the proof. Rather, we used it indirectly to guarantee the existence of the Picard scheme $\mathrm{Pic}_{\bar{J}/S}$. Thus, the lemma is valid whenever this Picard scheme exists, for example, when S is the spectrum of a field; see Corollaire 1.2 on p. 596 in [**18**].

Remark (2.4). Under the conditions of Proposition (2.2), assume that there is an invertible sheaf $\mathcal{L}$ of degree 1 on C/S. Then the map $\beta\colon J \to \mathrm{Pic}_{\bar{J}/S}$ can be constructed in another and more traditional way than that used in the proof of the

lemma. Namely, β can be constructed using the theta divisor associated to $\mathcal{L}$. This is a divisor $\Theta_{\mathcal{L}}$ on $\bar{J}$, and it may be constructed as follows.

Use the notation of the proof of the proposition. In addition, let g denote the (locally constant) arithmetic genus of the fibers of C/S. Now, on each fiber of the projection $p_{23}: C \times \bar{J} \times S' \to \bar{J} \times S'$, the restriction of $\mathcal{I} \otimes p_1^*\mathcal{L}^{\otimes g-1}$ has Euler characteristic 0. It follows that, on $\bar{J} \times S'$, the invertible sheaf

$$\mathcal{D}_{p_{23}}(\mathcal{I} \otimes p_1^*\mathcal{L}^{\otimes g-1})$$

has a canonical regular section; denote its divisor of zeros by $\Theta'_{\mathcal{L}}$. Arguing as in the proof of the lemma, we can show that $\Theta'_{\mathcal{L}}$ descends to a divisor $\Theta_{\mathcal{L}}$ on $\bar{J}$.

Let $\tau: \bar{J} \times J \to \bar{J}$ be the multiplication map; it is defined by $p_{124}^*\mathcal{I} \otimes p_{134}^*\mathcal{M}$ on $C \times \bar{J} \times J \times S'$. On $\bar{J} \times J \times S'$, consider $\mathcal{M}^\diamond$, and on $\bar{J} \times J$, form the sheaf,

$$\mathcal{T} := \mathcal{O}_{\bar{J}\times J}(p_1^*\Theta_{\mathcal{L}} - \tau^*\Theta_{\mathcal{L}}).$$

We are about to construct a faithfully flat covering S''/S' such that the pullbacks of $\mathcal{M}^\diamond$ and $\mathcal{T}$ are equal. Each sheaf defines a map from J to $\operatorname{Pic}_{\bar{J}/S}$, and these two maps are equal after we change the base to S''. So, by descent theory, the two maps are equal to begin with. Therefore, since $\mathcal{M}^\diamond$ defines β, so does $\mathcal{T}$.

To construct S''/S', we may replace S by S', and so assume $S' = S$. Moreover, we may assume that S is affine, so Noetherian, and is connected. After a further replacement of S, we may assume that the smooth locus C^{sm} of C/S admits a section σ; in fact, if we replace S by C^{sm}, then the diagonal provides the desired section σ. Fix m so large that $\mathcal{L}(m\sigma(S))$ is very ample. Then, again after replacing S, we can find a hyperplane section H of C, which is flat over S and whose support lies in C^{sm}.

Given any relative effective divisor H_0 on C^{sm}/S of relative degree n, we can find a faithfully flat covering of S such that, after replacing S, we can find sections σ_i of C^{sm}/S such that

$$H_0 = \sigma_1(S) + \ldots + \sigma_n(S). \tag{2.4.1}$$

Indeed, H_0/S is a faithfully flat covering, and after replacing S by H_0, we have a canonical section σ_1 of C^{sm}/S whose image is a subscheme of H_0 (in fact, σ_1 is simply the diagonal map of the original H_0/S). Form

$$H_1 := H_0 - \sigma_1(S).$$

It is a relative effective divisor on C^{sm}/S of constant relative degree $n-1$. Hence, by induction, we may assume that, after replacing S, we can find sections $\sigma_2, \ldots, \sigma_n$ of C^{sm}/S such that $H_1 = \sigma_2(S) + \ldots + \sigma_n(S)$. Then (2.4.1) holds. Taking H_0 to be H, we conclude that we may assume that we have sections σ_i of C^{sm}/S such that

$$\mathcal{L} = \mathcal{O}_C(T), \text{ where } T := \sigma_1(S) + \ldots + \sigma_{m+1}(S) - m\sigma(S).$$

Given a Cartier divisor D on C, set $\mathcal{I}(D) := \mathcal{I} \otimes p_1^*\mathcal{O}_C(D)$ and

$$\mathcal{M}^\diamond[D] := \big(\mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}(D) \otimes p_{13}^*\mathcal{M})\big)^{-1} \otimes \mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}(D)).$$

Then $\mathcal{M}^\diamond = \mathcal{M}^\diamond[0]$, and $\mathcal{T} = \mathcal{M}^\diamond[(g-1)T]$. Hence, it now suffices to prove the following assertion: given any section ρ of C^{sm}/S, set $R := \rho(S)$ and let $E := D+R$;

then $\mathcal{M}^\diamond[D]$ and $\mathcal{M}^\diamond[E]$ differ by tensor product with the pullback of an invertible sheaf on J.

To prove this assertion, consider the natural exact sequence,

$$0 \to \mathcal{O}_C(-R) \to \mathcal{O}_C \to \mathcal{O}_R \to 0.$$

Pull it back to $C \times \bar{J} \times J$, then tensor with $p_{12}^*\mathcal{I}(E) \otimes p_{13}^*\mathcal{M}$ and with $p_{12}^*\mathcal{I}(E)$. Identify $R \times \bar{J}$ with $\bar{J}$. Additivity of the determinant of cohomology now yields

$$\begin{aligned}\mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}(E) \otimes p_{13}^*\mathcal{M}) =& \mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}(D) \otimes p_{13}^*\mathcal{M}) \otimes p_1^*(\mathcal{I}(E)|\bar{J}) \otimes p_2^*(\mathcal{M}|J)\\ \mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}(E)) =& \mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}(D)) \otimes p_1^*(\mathcal{I}(E)|\bar{J}).\end{aligned}$$

Hence $\mathcal{M}^\diamond[D]$ and $\mathcal{M}^\diamond[E]$ differ by tensor product with $p_2^*(\mathcal{M}|J)$. So the assertion holds. Thus $\mathcal{T}$ defines $\beta: J \to \mathrm{Pic}_{\bar{J}/S}$.

3. Theorem of the cube

(3.1) *Abel maps.* Let C/S be a flat projective family of integral curves, m and n integers. The *Abel map* of bidegree (m, n) is defined to be the map,

$$A_{C/S}: \mathrm{Hilb}^m_{C/S} \times J^n_{C/S} \to \bar{J}^{n-m}_{C/S},$$

given by tensoring the ideal of an m-cluster with a degree-n invertible sheaf. We will often abbreviate $A_{C/S}$ by A.

More precisely, an S-map $t: T \to \mathrm{Hilb}^m_{C/S} \times J^n$ corresponds to a pair consisting of a flat closed subscheme Y of $C \times T/T$ with length-m fibers and of an invertible sheaf $\mathcal{L}'$ on $C \times T'/T'$ with degree-n fibers, where T'/T is an étale covering, such that the two pullbacks of $\mathcal{L}'$ to $C \times T''$ are equal, where $T''\big/ T' \times_T T'$ is an étale covering. Let $\mathcal{I}'$ denote the ideal of $Y \times T'$. Then $\mathcal{I}' \otimes \mathcal{L}'$ is a torsion-free rank-1 sheaf of degree $n - m$ on $C \times T'/T'$, and its two pullbacks to $C \times T''$ are equal. Hence $\mathcal{I}' \otimes \mathcal{L}'$ defines a map $A(t): T \to \bar{J}^{n-m}$.

The Abel map is smooth when the geometric fibers of C/S have double points at worst, thanks to the following more general fact.

(Smoothness) *If all the fibers of C/S are Gorenstein, then the Abel map A is smooth.*

This fact is proved in Corollary (2.6) of [**9**], as an application of an even more general statement, Theorem (2.4) of [**9**].

Lemma (3.2). *Let C/S be a flat projective family of integral curves. Assume there is a universal sheaf $\mathcal{I}$ on $C \times \bar{J}^1$. Set $P := \mathbf{P}(\mathcal{I})$. Let $Z \subset P$ be the preimage of $C \times J^1$. Then the structure map of P induces an isomorphism $Z \xrightarrow{\sim} C \times J^1$, and there is a map $\zeta: P \to \bar{J}$ extending the Abel map $A: C \times J^1 \to \bar{J}$.*

Proof. Let $\rho: P \to C \times \bar{J}^1$ be the structure map; say $\rho = (\rho_1, \rho_2)$. Set $\gamma := (\rho_1, 1_P)$ and $\theta := 1_C \times \rho_2$. Then ρ factors as follows:

$$\rho: P \xrightarrow{\gamma} C \times P \xrightarrow{\theta} C \times \bar{J}^1.$$

So there is a natural isomorphism $\gamma^*\theta^*\mathcal{I} \xrightarrow{\sim} \rho^*\mathcal{I}$. Let $q: \theta^*\mathcal{I} \to \gamma_*\rho^*\mathcal{I}$ be its adjoint. In other words, γ is the graph map of ρ_1, and its image, Y say, is the graph subscheme; in these terms, q is equal to the natural quotient map $\theta^*\mathcal{I} \twoheadrightarrow \theta^*\mathcal{I}|Y$.

Let $u\colon \rho^*\mathcal{I} \to \mathcal{O}_P(1)$ be the universal map, and form the composition,

$$r\colon \theta^*\mathcal{I} \xrightarrow{q} \gamma_*\rho^*\mathcal{I} \xrightarrow{\gamma_* u} \gamma_*\mathcal{O}_P(1).$$

Then r is a surjective map between P-flat sheaves on $C \times P$. Set $\mathcal{J} := \mathcal{K}er(r)$. Then $\mathcal{J}$ is flat too, and forming it commutes with passing to the fibers. Hence $\mathcal{J}$ is a torsion-free rank-1 sheaf of degree 0 on $C \times P$. So it defines a map $\zeta\colon P \to \bar{J}$.

Since $\mathcal{I}|C \times J^1$ is invertible, ρ restricts to an isomorphism $Z \xrightarrow{\sim} C \times J^1$, and $u\colon \rho^*\mathcal{I} \to \mathcal{O}_P(1)$ restricts to an isomorphism. Hence the exact sequence,

$$0 \longrightarrow \mathcal{J} \longrightarrow \theta^*\mathcal{I} \xrightarrow{r} \gamma_*\mathcal{O}_P(1) \longrightarrow 0,$$

is equal, on $C \times Z$, to the tensor product of $\theta^*\mathcal{I}$ with the basic sequence,

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_{C\times P} \longrightarrow \mathcal{O}_Y \longrightarrow 0,$$

where $\mathcal{I}_Y$ is the ideal of the graph subscheme Y. However, this sequence is the pullback under $1_C \times \rho_1\colon C \times P \to C \times C$ of the basic sequence of the diagonal, (2.2.2). Hence $\mathcal{J}|C \times Z$ defines $A \circ (\rho|Z)$. Thus ζ extends $A\colon C \times J^1 \to \bar{J}$, and the proof is complete.

Remark (3.3). In the proof of Proposition (3.2), consider the map r. Being surjective, r defines a map,

$$P \to \mathrm{Quot}^1_{\mathcal{I}/C\times \bar{J}^1/\bar{J}^1},$$

and it is not hard to see that this map is an isomorphism. (In fact, there is nothing special about $\mathcal{I}/C \times \bar{J}^1/\bar{J}^1$. This is a simple general phenomenon. See [**12**, (2.2), p. 109].) Thus $\zeta\colon P \to \bar{J}$ is a universal lifting of $A\colon C \times J^1 \to \bar{J}$ in the following sense: given a map $T \to \bar{J}$ defined by a degree-0 torsion-free rank-1 subsheaf of a degree-1 torsion-free rank-1 sheaf on $C \times T$, there exists a unique map $T \to P$ such that the pair of sheaves is the pullback of the pair consisting of $\mathcal{J}$ and $\theta^*\mathcal{I}$.

Lemma (3.4). *Let C/S be a flat projective family of integral curves, and $T \to C$ an S-map. Let Γ be the graph subscheme of $C\times T$, and $\mathcal{I}_\Gamma$ its ideal. Set $W := \mathbf{P}(\mathcal{I}_\Gamma)$, and let $\psi\colon W \to C \times T$ be the structure map. Assume that the geometric fibers of C/S only have surficial singularities. Then W/T is flat, and*

$$\psi_*\mathcal{O}_W = \mathcal{O}_{C\times T} \text{ and } R^i\psi_*\mathcal{O}_W = 0 \text{ for } i \geq 1. \tag{3.4.1}$$

Proof. Without loss of generality, we may replace S by $\operatorname{Spec}\mathcal{O}_{S,s}$ where s is an arbitrary point of S. By [**11**, $\mathrm{O_{III}}$ 10.3.1, p. 20], there exists a flat local $\mathcal{O}_{S,s}$-algebra whose residue field is any given extension of the field of s, and we may replace S by the spectrum of this algebra. Thus we may assume that S is a local scheme with closed point s whose residue field $k(s)$ is algebraically closed.

Embed C/S in a projective space $\mathbf{P}^N_S$ for some N. Let $\mathcal{H}$ be the ideal of C. Since C/S is flat, $\mathcal{H}(s)$ is the ideal of $C(s)$, and $\mathcal{H}$ is flat. Also, for $m \gg 0$, the base change map is an isomorphism,

$$H^0(\mathcal{H}(m)) \otimes k(s) \xrightarrow{\sim} H^0(\mathcal{H}(s)(m)).$$

Fix $m \gg 0$, and take $N-2$ general sections of $\mathcal{H}(s)(m)$. Via the above isomorphism, lift the sections back to sections of $\mathcal{H}(m)$, and form the scheme F of

common zeros of the lifts. Then $F \supset C$. Moreover, increasing m if necessary, we may assume that $F(s)$ is a smooth surface since every singularity of C is surficial (see the proofs of (7)–(9) in [**3**] for example). Hence F/S is a smooth family of surfaces.

Consider the nested sequence of subschemes,

$$\Gamma \subset C \times T \subset F \times T.$$

Since F/S is smooth and Γ is a graph, Γ is regularly embedded in $F \times T$, say with ideal $\mathcal{J}$. Hence the symmetric algebra of $\mathcal{J}$ is equal to its Rees algebra by Micali's theorem [**15**, p. 1955]. So $\mathbf{P}(\mathcal{J})$ is equal to the blowup B of $F \times T$ along Γ. Since $F \times T/T$ and Γ/T are both smooth, so is B/T.

Denote the sheaf of ideals of $C \times T$ in $F \times T$ by $\mathcal{K}$, and form the exact sequence,

$$0 \to \mathcal{K} \to \mathcal{J} \to \mathcal{I}_\Gamma \to 0.$$

It gives rise to the following exact sequence of sheaves of graded modules over the symmetric algebra $Sym(\mathcal{J})$ (see [**7**, p. 571]):

$$0 \to \mathcal{K} \cdot Sym(\mathcal{J})[-1] \to Sym(\mathcal{J}) \to Sym(\mathcal{I}_\Gamma) \to 0.$$

Taking associated sheaves yields the following exact sequence on B:

$$0 \to \mathcal{K} \cdot \mathcal{O}_B(-1) \to \mathcal{O}_B \to \mathcal{O}_W \to 0. \tag{3.4.2}$$

Thus $\mathcal{K} \cdot \mathcal{O}_B(-1)$ is the ideal of W on B.

Since F/S is smooth and C/S is flat, C is a Cartier divisor on F. Hence $\mathcal{K}$ is invertible. Thus W is a Cartier divisor on B; in fact, $W = D - E$ where D is the preimage of $C \times T$ in B and where E is the exceptional divisor. Since W remains a Cartier divisor on the fibers of B/T and since B/T is flat, W/T is flat.

Consider the blowup map $\beta\colon B \to F \times T$. Since $\mathcal{K}$ is invertible, the projection formula yields, for every i,

$$R^i\beta_*\mathcal{K} \cdot \mathcal{O}_B(-1) = \mathcal{K} \cdot R^i\beta_*\mathcal{O}_B(-1).$$

Hence Assertion (3.4.1) follows from the long exact sequence of higher direct images of β associated to the sequence (3.4.2) and from the following formulas:

$$\beta_*\mathcal{O}_B(n) = \mathcal{J}^n \text{ and } R^i\beta_*\mathcal{O}_B(n) = 0 \text{ for } i \geq 1 \text{ and } n \geq -1, \tag{3.4.3}$$

where $\mathcal{J}^n = \mathcal{O}_{F\times T}$ for $n \leq 0$ by convention. These formulas are proved next (and the proof applies more generally to any blowup along a regularly embedded center of codimension at least 2).

Since $B = \mathbf{P}(\mathcal{J})$, restricting the base yields $E = \mathbf{P}(\mathcal{J}/\mathcal{J}^2)$. Now, $\mathcal{J}/\mathcal{J}^2$ is locally free. Hence $R^i\beta_*\mathcal{O}_E(n) = 0$ for $i \geq 1$ and $n \geq -1$ by Serre's computation. So the long exact sequence associated to the sequence,

$$0 \to \mathcal{O}_B(n+1) \to \mathcal{O}_B(n) \to \mathcal{O}_E(n) \to 0, \tag{3.4.4}$$

yields a surjection $R^i\beta_*\mathcal{O}_B(n+1) \twoheadrightarrow R^i\beta_*\mathcal{O}_B(n)$ for $i \geq 1$ and for $n \geq -1$. By Serre's theorem, $R^i\beta_*\mathcal{O}_B(n)$ vanishes for $i \geq 1$ and $n \gg 0$. Hence, by descending induction on n, it vanishes for $i \geq 1$ and $n \geq -1$.

Sequence (3.4.4) also gives rise to the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J}^{n+1} & \longrightarrow & \mathcal{J}^{n} & \longrightarrow & \mathcal{J}^{n}/\mathcal{J}^{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \beta_*\mathcal{O}_B(n+1) & \longrightarrow & \beta_*\mathcal{O}_B(n) & \longrightarrow & \beta_*\mathcal{O}_E(n) & &
\end{array}$$

Since $E = \mathbf{P}(\mathcal{J}/\mathcal{J}^2)$, the right vertical map is an isomorphism for $n \ge -1$ by Serre's computation again. The left vertical map is an isomorphism for $n \gg 0$ by Serre's theorem. Hence, by descending induction on n, the middle vertical map is an isomorphism for $n \ge -1$. Thus Formulas (3.4.3) hold, and the proof is complete.

Lemma (3.5) (Generalized theorem of the cube). *Let S be a connected and locally Noetherian scheme. Let $g\colon Y \to S$ and $f\colon X \to Y$ be flat and proper maps. Let $\sigma\colon S \to Y$ and $\tau\colon Y \to X$ be sections of g and f. Assume*

(i) *that $\mathcal{O}_S = g_*\mathcal{O}_Y$ and $\mathcal{O}_Y = f_*\mathcal{O}_X$ hold universally, and*

(ii) *that, for every closed point $s \in S$, the natural map on the fibers is injective:*

$$w\colon H^0(Y(s), R^1f(s)_*\mathcal{O}_{X(s)}) \hookrightarrow H^1(f(s)^{-1}\sigma(s), \mathcal{O}_{f(s)^{-1}\sigma(s)}).$$

Let $s_0 \in S$ and $\mathcal{L}$ be an invertible sheaf on X. If the three restrictions,

$$\mathcal{L}|X(s_0),\ \mathcal{L}|\tau(Y),\ \text{and}\ \mathcal{L}|f^{-1}\sigma(S),$$

are trivial, then $\mathcal{L}$ is trivial.

Proof. It is not hard to adapt, mutatis mutandis, Mumford's proof of his similar theorem on p. 91 in [**17**]. It is a straightforward job, except at the beginning and at the end. At the beginning, Mumford uses his proposition on p. 89 to obtain the existence of a maximum closed subscheme T of S carrying an invertible sheaf $\mathcal{N}$ such that $h^*\mathcal{N} = \mathcal{L}|h^{-1}T$ where $h := gf$. (In fact, forming T commutes with base-changing S.) Mumford's construction of T is not hard to adapt; here's the idea.

Since h is flat and proper, by [**11**, III_2 7.7.6, p. 69], there are coherent sheaves $\mathcal{M}$ and $\mathcal{N}$ on S such that, for every coherent sheaf $\mathcal{F}$ on S, we have

$$h_*(\mathcal{L} \otimes h^*\mathcal{F}) = \mathcal{H}om(\mathcal{M}, \mathcal{F}) \text{ and } h_*(\mathcal{L}^{-1} \otimes h^*\mathcal{F}) = \mathcal{H}om(\mathcal{N}, \mathcal{F}).$$

Take $T := \mathrm{Supp}(\mathcal{M}) \bigcap \mathrm{Supp}(\mathcal{N})$ using the annihilators of $\mathcal{M}$ and $\mathcal{N}$ to define the scheme structures on their supports.

Over a point $t \in T$, the restrictions $\mathcal{L}|X(t)$ and $\mathcal{L}^{-1}|X(t)$ each have a nonzero section. Compose the first section with the dual of the second, obtaining a nonzero map $\mathcal{O}_{X(t)} \to \mathcal{O}_{X(t)}$. This map is given by multiplication by a scalar because $h_*\mathcal{O}_X = \mathcal{O}_T$ holds universally. Hence $\mathcal{L}|X(t)$ is trivial. Therefore, on a neighborhood of t, one element suffices to generate $\mathcal{N}$. It follows that $\mathcal{N}|T$ is an invertible $\mathcal{O}_T$-module and that $h^*\mathcal{N}|h^{-1}T = \mathcal{L}|h^{-1}T$.

At the end of the proof of his theorem, Mumford uses the Künneth formula. Instead, we must use a related injection, which we get as follows. For each closed point $s \in S$, form the exact sequence of terms of low degree of the Leray spectral sequence:

$$H^1(Y(s), f(s)_*\mathcal{O}_{X(s)}) \xrightarrow{u} H^1(X(s), \mathcal{O}_{X(s)}) \xrightarrow{v} H^0(Y(s), R^1f(s)_*\mathcal{O}_{X(s)}).$$

Since $f(s)_*\mathcal{O}_{X(s)} = \mathcal{O}_{Y(s)}$ by Assumption (i), the section $\tau(s): Y(s) \to X(s)$ of $f(s)$ yields a map,

$$u': H^1(X(s), \mathcal{O}_{X(s)}) \to H^1(Y(s), \mathcal{O}_{Y(s)}),$$

splitting u. Hence, by Assumption (ii), the map $(u', w \circ v)$ is an injection,

$$H^1(X(s), \mathcal{O}_{X(s)}) \hookrightarrow H^1(Y(s), \mathcal{O}_{Y(s)}) \bigoplus H^1(f(s)^{-1}\sigma(s), \mathcal{O}_{f(s)^{-1}\sigma(s)}).$$

This injection works in place of the Künneth formula.

Lemma (3.6). *Let C/S be a flat projective family of integral curves with surficial singularities, and let $\mathcal{P}$ be an invertible sheaf on $\bar{J}$. Assume that S is connected, and that S contains a point s_0 such that the fiber $\mathcal{P}(s_0)$ is trivial. Form the Abel pullback $A^*\mathcal{P}$ on $C \times J^1$. Then the following two assertions hold.*

(1) *The pullback $A^*\mathcal{P}$ defines a map $a: J^1 \to J$, and a factors uniquely through the structure map $J^1 \to S$.*

(2) *If the smooth locus C^{sm} of C/S admits a section, then there are invertible sheaves $\mathcal{M}_1$ on C and $\mathcal{M}_2$ on J^1 such that*

$$A^*\mathcal{P} = p_1^*\mathcal{M}_1 \otimes p_2^*\mathcal{M}_2,$$

where the p_i are the projections.

Proof. Consider Part (1). A priori, $A^*\mathcal{P}$ defines a map from J^1 to $\mathrm{Pic}_{C/S}$. However, the image lies in the open subscheme J because $A^*\mathcal{P}(s_0)$ is trivial and S is connected.

If there is an S-factorization $a: J^1 \xrightarrow{b} S \xrightarrow{c} J$, then b must be the structure map. So b is faithfully flat. Hence, by descent theory, c is uniquely determined.

Assume for a moment that the smooth locus C^{sm} of C/S admits a section, and that Part (2) holds. Then $\mathcal{M}_1$ defines an S-map $c: S \to J$ such that $a = cb$, where b is the structure map.

In general, there is an étale covering S'/S such that $C^{\mathrm{sm}} \times S'/S'$ admits a section. So, by the reasoning above, $a \times S'$ factors through a unique map $c': S' \to J \times S'$. Set $S'' := S' \times S'$. Then $a \times S''$ factors through both $c' \times S'$ and $S' \times c'$. So the latter two maps are equal. Hence c' descends to a suitable map $c: S \to J$. Thus Part (1) follows from Part (2).

To prove Part (2), assume from now on that C^{sm} admits a section. Then there exists a universal sheaf $\mathcal{I}$ on $C \times \bar{J}^1$. Set $P := \mathbf{P}(\mathcal{I})$. Let $\rho: P \to C \times \bar{J}^1$ denote the structure map, and

$$\rho_1: P \to C \text{ and } \rho_2: P \to \bar{J}^1$$

the natural maps. Now, $A: C \times J^2 \to \bar{J}^1$ is smooth; see (3.1). So its image V is open. Also $V \supset J^1$. Set

$$X := \rho^{-1}(C \times V) \text{ and } Z := \rho^{-1}(C \times J^1).$$

Then ρ induces an isomorphism $Z \xrightarrow{\sim} C \times J^1$; so the ρ_i extend the projections p_i.

Say $\theta: S \to C$ is the section, let Q be its image, and set $\mathcal{L} := \mathcal{O}_C(Q)$. Then $\mathcal{O}_C(Q)$ defines a section $\phi: S \to \bar{J}^1$, whose image lies in J^1. So ϕ defines a section $\xi_1: C \to P$ of ρ_1 because ρ is an isomorphism over $C \times J^1$. Moreover, θ defines a section $\xi_2: \bar{J}^1 \to P$ of ρ_2 because ρ is an isomorphism over $C^{\mathrm{sm}} \times \bar{J}^1$.

Consider the map $\zeta\colon P \to \bar{J}$ of (3.2) extending $A\colon C \times J^1 \to \bar{J}$. Set

$$\begin{aligned} \mathcal{Q}_2 &:= \zeta^*\mathcal{P}, & \mathcal{N}_2 &:= \xi_2^*\mathcal{Q}_2; \\ \mathcal{Q}_1 &:= \mathcal{Q}_2 \otimes \rho_2^*\mathcal{N}_2^{-1}, & \mathcal{N}_1 &:= \xi_1^*\mathcal{Q}_1; \\ \mathcal{Q}_0 &:= \mathcal{Q}_1 \otimes \rho_1^*\mathcal{N}_1^{-1} = \mathcal{Q}_2 \otimes \rho_2^*\mathcal{N}_2^{-1} \otimes \rho_1^*\mathcal{N}_1^{-1}. \end{aligned}$$

Notice that, by hypothesis and by construction, the restrictions,

$$\mathcal{Q}_0|P(s_0) \text{ and } \mathcal{Q}_0|\xi_2(\bar{J}^1) \text{ and } \mathcal{Q}_0|\rho_2^{-1}\phi(S), \tag{3.6.1}$$

are trivial. It suffices to prove that $\mathcal{Q}_0|X$ is trivial.

Set $q := \rho_2|X$, so $q\colon X \to V$. Then q is flat. Indeed, let Δ be the diagonal subscheme of $C \times C$, and $\mathcal{I}_\Delta$ its ideal. Set $W := \mathbf{P}(\mathcal{I}_\Delta)$. Then there is a Cartesian square

$$\begin{tikzcd} X \arrow[d] & & W \times J^2 \arrow[ll] \arrow[d] \\ C \times V \arrow[rr, phantom, "\square", near start, shift left=4ex] & & C \times C \times J^2 \arrow[ll, "1\times A"'] \end{tikzcd}$$

because, on $C \times C \times J^2$, the pullback of $\mathcal{I}|C \times V$ is equal to the tensor product of the pullback of $\mathcal{I}_\Delta$ with an invertible sheaf (namely, the pullback of a universal sheaf on $C \times J^2$). So there is a Cartesian square

$$\begin{tikzcd} X \arrow[d, "q"] \arrow[dr, phantom, "\square"] & W \times J^2 \arrow[l] \arrow[d] \\ V & C \times J^2 \arrow[l, "A"'] \end{tikzcd}$$

The right vertical map is flat thanks to Lemma (3.4) applied with C for T. Also, $A\colon C \times J^2 \to V$ is faithfully flat, being smooth and surjective. Hence q is flat.

Similarly, $q_*\mathcal{O}_X = \mathcal{O}_V$ holds universally. Indeed, by the preceding argument, this statement results from the corresponding statement about the composition $W \to C \times C \to C$. The corresponding statement results from Lemma (3.4) and a basic fact: $p_*\mathcal{O}_C = \mathcal{O}_S$ holds universally where $p\colon C \to S$ is the structure map.

The generalized theorem of the cube, Lemma (3.5), does not apply to the triple $X/V/S$ because V/S is not proper. However, V is swept out by copies of C because $A\colon C \times J^2 \to V$ is surjective. So we can circumvent the obstacle as follows.

Since $q\colon X \to V$ is flat and proper, and since $q_*\mathcal{O}_X = \mathcal{O}_V$ holds universally, there exists a maximum closed subscheme Y of V carrying an invertible sheaf $\mathcal{H}$ such that $q^*\mathcal{H} = \mathcal{Q}_0|q^{-1}Y$, and forming Y commutes with base-changing V; see the proof of Lemma (3.5). In fact, $\mathcal{H}$ is trivial because $\mathcal{Q}_0|\xi_2(V)$ is trivial. Hence $\mathcal{Q}_0|X$ is trivial if $Y = V$. Since forming Y commutes with base-changing V, it suffices to construct a faithfully flat map $U \to V$ such that the pullback of $\mathcal{Q}_0$ to $X \times_V U$ is trivial.

Given $n \geq 0$, let C_n denote the n-fold self-product of C^{sm}. Let $\gamma_n\colon C_n \to J^2$ be the map that sends n T-points of C^{sm}, say with graph images $Q_1, \ldots, Q_n$, to the T-point of J^2 representing the following invertible sheaf on $C \times T$:

$$\mathcal{O}((n+2)Q \times T - Q_1 - \cdots - Q_n).$$

Finally, set $\delta_n := A \circ (1_C \times \gamma_n)$, so $\delta_n\colon C \times C_n \to V$.

Consider the factorization $\gamma_n\colon C_n \xrightarrow{b} \mathrm{Hilb}^n_{C^{\mathrm{sm}}/S} \xrightarrow{c} J^2$. The map b is faithfully flat for every n, and c is faithfully flat for $n \gg 0$. So γ_n is faithfully flat for $n \gg 0$. Hence, δ_n is faithfully flat for $n \gg 0$, since $A\colon C \times J^2 \to V$ is faithfully flat.

Thus, it suffices to prove that the pullback of $\mathcal{Q}_0$ to $X \times_V (C \times C_n)$ is trivial for every n. To do so, we apply Lemma (3.5) to $X \times_V (C \times C_n)/(C \times C_n)/C_n$ with the sections,

$$\sigma := \theta \times 1_{C_n} \text{ and } \tau := \xi_2 \times 1_{C \times C_n}.$$

Assumption (i) of (3.5) holds as $q_*\mathcal{O}_X = \mathcal{O}_V$ and $p_*\mathcal{O}_C = \mathcal{O}_S$ hold universally.

To check Assumption (ii), let $t \in C_n$. Set $T := C \otimes k(t)$ and $W_T := W \otimes k(t)$. Denote by $\psi\colon W_T \to C \times T$ the structure map, and by $\psi_2\colon C \times T \to T$ the second projection. Then Lemma (3.4) implies that $\psi_*\mathcal{O}_{W_T} = \mathcal{O}_{C\times T}$ and $R^1\psi_*\mathcal{O}_{W_T} = 0$. Hence, using the Leray spectral sequence and making the substitution, we get

$$R^1(\psi_2\psi)_*\mathcal{O}_{W_T} = R^1\psi_{2*}(\psi_*\mathcal{O}_{W_T}) = R^1\psi_{2*}\mathcal{O}_{C\times T}.$$

Now, $C \times T = T \times T$. So, commuting cohomology with flat base change, we get

$$R^1\psi_{2*}\mathcal{O}_{C\times T} = H^1(T, \mathcal{O}_T) \otimes \mathcal{O}_T.$$

Therefore, the following natural map is an isomorphism:

$$H^0(T,\, R^1(\psi_2\psi)_*\mathcal{O}_{W_T}) \xrightarrow{\sim} H^1(T, \mathcal{O}_T).$$

It now follows immediately that Assumption (ii) holds.

It remains to check the triviality of the three appropriate pullbacks of $\mathcal{Q}_0$. First, C_n is connected and maps onto S because the fibers of C/S are geometrically integral. Moreover, $\mathcal{Q}_0|P(s_0)$ is trivial by (3.6.1). Hence, C_n contains a point t_0 that maps to s_0, and the pullback of $\mathcal{Q}_0$ to $X \otimes k(t_0)$ is trivial.

Second, consider the pullback of $\mathcal{Q}_0$ to $\tau(C\times C_n)$. This pullback is trivial because $\mathcal{Q}_0|\xi_2(\bar{J}^1)$ is trivial by (3.6.1).

Finally, consider the pullback of $\mathcal{Q}_0$ to $X \times_V \sigma(C_n)$. To begin, suppose $n = 0$. Now, $C_0 = S$. So $\sigma = \theta$ and $\delta_0\theta = \phi$. Hence $X \times_V \sigma(C_0)$ is equal to $\rho_2^{-1}\phi(S)$. However, the pullback of $\mathcal{Q}_0$ to the latter scheme is trivial by (3.6.1).

Proceeding by induction on n, suppose that the pullback of $\mathcal{Q}_0$ to $X \times_V \sigma(C_n)$ is trivial. Then Lemma (3.5) implies that the pullback of $\mathcal{Q}_0$ to $X \times_V (C \times C_n)$ is trivial. Hence so is the pullback to $X \times_V (C^{\mathrm{sm}} \times C_n)$. However, the latter scheme is equal to $X \times_V \sigma(C_{n+1})$, as is easy to see. The proof of the lemma is now complete.

Proposition (3.7). *Let C/S be a flat projective family of integral curves with surficial singularities. Assume S is connected, and let U be the connected component of* $\mathrm{Pic}_{\bar{J}/S}$ *containing the zero section. Then there exists a natural map $c\colon U \to J$ such that $c \circ \beta = 1_J$, where β is the map of Proposition* (2.2). *Furthermore, given any invertible sheaf $\mathcal{L}$ of degree* 1 *on C/S, the map $A^*_{\mathcal{L}}$ of Subsection* (2.1) *restricts to c; in particular, $A^*_{\mathcal{L}}|U$ is independent of $\mathcal{L}$.*

Proof. The structure sheaf $\mathcal{O}_C$ defines a section of $\bar{J}/S$. Hence, $\bar{J} \times U$ admits a universal invertible sheaf $\mathcal{P}$; see Prop. 2.1 on Page 232-04 of [**10**]. Also, for every point u_0 of U on the identity section, $\mathcal{P}(u_0)$ is trivial. On $C \times J^1 \times U$, form

$(A \times 1_U)^*\mathcal{P}$. This pullback defines a map $a: J^1 \times U \to J$. It factors through a map $c: U \to J$ by virtue of Part (1) of Lemma (3.6) applied to $C \times U/U$ and $\mathcal{P}$.

Given $\mathcal{L}$, let $[\mathcal{L}] \in J^1(S)$ represent it. Then the fiber $a([\mathcal{L}]): U \to J$ is equal to c on the one hand, and to $A_{\mathcal{L}}^*|U$ on the other. Thus the last assertion holds.

The equation $c \circ \beta = 1_J$ follows. Indeed, it suffices to check this equation after making an étale base change S'/S. After a suitable such base change, there exists a $\mathcal{L}$. Then, by what we just proved, c is equal to $A_{\mathcal{L}}^*$, and so Proposition (2.2) yields the asserted equation. The proof is now complete.

4. Proof and extension

(4.1) *Proof of the autoduality theorem of* (2.1). For a moment, make the following assumption: for each geometric point s of S and for some invertible sheaf $\mathcal{L}_s$ of degree 1 on the curve $C(s)$, the Abel map induces an isomorphism,

$$A_{\mathcal{L}_s}^*: \operatorname{Pic}^\tau_{\bar{J}(s)} \xrightarrow{\sim} J(s).$$

This case of the theorem implies the general case, as we'll now prove.

Set $U := \operatorname{Pic}^\tau_{\bar{J}/S}$ and consider the map $\beta: J \to \operatorname{Pic}_{\bar{J}/S}$ of Proposition (2.2). The image of β lies in $\operatorname{Pic}^0_{\bar{J}/S}$, hence also in U. Since $A_{\mathcal{L}}^* \circ \beta = 1_J$ for any invertible sheaf $\mathcal{L}$ of degree 1 on C/S, we have to prove that $\beta: J \to U$ is an isomorphism. To do so, we may change the base to an étale covering of S. Thus we may assume that the smooth locus of C/S admits a section. Then C/S does carry an invertible sheaf $\mathcal{L}$ of degree 1. Hence $\beta: J \to U$ is a right inverse, so a closed embedding. Since J is flat over S, it follows that $\beta: J \to U$ is an isomorphism, being one on each geometric fiber. Therefore, $U = \operatorname{Pic}^0_{\bar{J}/S}$. Thus to prove the theorem, we may assume that S is the spectrum of an algebraically closed field.

Proceed by induction on the difference δ between the arithmetic genus and the geometric genus of C. First, assume $\delta = 0$. Then C is smooth. Hence J is complete, so an Abelian variety. Given any Abelian variety G, in the theorem on p. 125 of [**17**], Mumford proves that the scheme Pic^0_G is a quotient of G by a finite group; hence, Pic^0_G is integral and has the same dimension as G. Moreover, Pic^0_G is equal to $\operatorname{Pic}^\tau_G$ by Corollary 2 on p. 178 of [**17**]. Now, $A_{\mathcal{L}}^* \circ \beta = 1_J$ for any invertible sheaf $\mathcal{L}$ of degree 1 on C by Proposition (2.2). So β is a closed embedding of J in Pic^0_J. Hence β is an isomorphism. Thus the theorem holds when $\delta = 0$.

Assume $\delta \geq 1$ from now on. Fix an invertible sheaf $\mathcal{L}$ of degree 1 on C. Then $A_{\mathcal{L}}^* \circ \beta = 1_J$ by Proposition (2.2). So $A_{\mathcal{L}}^*$ is an epimorphism. We must prove it is a monomorphism. So let $\phi: T \to U$ be an S-map of finite type, say arising from the invertible sheaf $\mathcal{N}$ on $\bar{J} \times T$. Assume that $(A_{\mathcal{L}} \times 1_T)^*\mathcal{N}$ is equal to the pullback of an invertible sheaf on T. We must prove that $\mathcal{N}$ is the pullback of an invertible sheaf on T. To do so, we may assume that T is connected.

First set $U^0 := \operatorname{Pic}^0_{\bar{J}}$ and assume $\phi(T) \subset U^0$. Note that U^0 is an open subscheme of U because we are now working over an algebraically closed field. Let $\mathcal{P}$ be a universal invertible sheaf on $\bar{J} \times U^0$. Then there is an invertible sheaf $\mathcal{T}$ on T such that

$$(1 \times \phi)^*\mathcal{P} = \mathcal{N} \otimes q_2^*\mathcal{T} \text{ on } \bar{J} \times T,$$

where $q_2\colon \bar J \times T \to T$ is the projection.

Let $u_0 \in U^0$ denote the identity; so $\mathcal{P}(u_0)$ is trivial. Hence Lemma (3.6) applies to $C \times U^0/U^0$ and $\mathcal{P}$. Part (2) of the lemma implies that $(A \times 1)^*\mathcal{P}$ is equal on $C \times J^1 \times U^0$ to the tensor product of the pullbacks of invertible sheaves on $C \times U^0$ and $J^1 \times U^0$. Therefore, there are invertible sheaves $\mathcal{N}_1$ on $J^1 \times T$ and $\mathcal{N}_2$ on $C \times T$ such that

$$(A \times 1)^*\mathcal{N} = q_{23}^*\mathcal{N}_1 \otimes q_{13}^*\mathcal{N}_2 \text{ on } C \times J^1 \times T,$$

where q_{23} and q_{13} are the projections.

Let $[\mathcal{L}] \in J^1$ represent $\mathcal{L}$. Then the equation above yields

$$(A_{\mathcal{L}} \times 1)^*\mathcal{N} = ((A \times 1)^*\mathcal{N})[\mathcal{L}] = (q_2^*\mathcal{N}_1[\mathcal{L}]) \otimes \mathcal{N}_2 \text{ on } C \times T.$$

By assumption, the term on the left is the pullback of an invertible sheaf on T; hence, so is $\mathcal{N}_2$. Therefore,

$$(A \times 1)^*\mathcal{N} = q_{23}^*\mathcal{R} \tag{4.1.1}$$

for some invertible sheaf $\mathcal{R}$ on $J^1 \times T$.

Since $\delta \geq 1$, there is a double point $Q \in C$. Let $\varphi\colon C^\dagger \to C$ be the blowup at Q. Then there is a natural scheme P, known as the *presentation scheme,* and there are natural maps $\kappa\colon P \to \bar J_C$ and $\pi\colon P \to \bar J_{C^\dagger}$; see Subsections (3.1) and (3.3), in [**9**], or [**6**]. Since Q is a double point, P is a $\mathbf{P}^1$-bundle over $\bar J_{C^\dagger}$; see Theorem (6.3) in [**9**]. Now, since $\mathcal{N}$ arises from a map $T \to U^0$, the pullback $(\kappa \times 1)^*\mathcal{N}$ restricts on each $\mathbf{P}^1$ to a sheaf of degree 0, hence is the pullback of an invertible sheaf $\mathcal{R}^\dagger$ on $\bar J_{C^\dagger} \times T$.

Set $\mathcal{L}^\dagger := \varphi^*\mathcal{L}$. By Lemma (6.4) in [**9**] the singularities of $C^\dagger$ are only double points. Hence, by Corollary (5.5) in [**9**], there is a map $\Lambda\colon C^\dagger \times J^1_C \to P$ making the following two diagrams commute:

$$\begin{array}{ccc} C^\dagger \times J^1_C & \xrightarrow{\Lambda} & P \\ {\scriptstyle 1\times\varphi^*}\downarrow & & \downarrow{\scriptstyle \pi} \\ C^\dagger \times J^1_{C^\dagger} & \xrightarrow{A_{C^\dagger}} & \bar J_{C^\dagger} \end{array} \qquad \begin{array}{ccc} C^\dagger \times J^1_C & \xrightarrow{\Lambda} & P \\ {\scriptstyle \varphi\times 1}\downarrow & & \downarrow{\scriptstyle \kappa} \\ C \times J^1_C & \xrightarrow{A_C} & \bar J_C \end{array} \tag{4.1.2}$$

Those diagrams yield these equations:

$$(A_{\mathcal{L}^\dagger} \times 1)^*\mathcal{R}^\dagger = (\Lambda_{\mathcal{L}} \times 1)^*(\kappa \times 1)^*\mathcal{N} = (\varphi \times 1)^*(A_{\mathcal{L}} \times 1)^*\mathcal{N} = p_2^*\mathcal{R}[\mathcal{L}],$$

where $p_2\colon C^\dagger \times T \to T$ is the projection, and $\Lambda_{\mathcal{L}}$ is the composition of Λ with the map $C^\dagger \to C^\dagger \times J^1_C$ defined by $\mathcal{L}$.

By induction, autoduality holds for $C^\dagger$. Hence $\mathcal{R}^\dagger$ is the pullback of an invertible sheaf on T, whence so is $(\kappa \times 1)^*\mathcal{N}$. Invert the latter sheaf on T, pull it back to $\bar J_C \times T$, tensor with $\mathcal{N}$, and use the product to replace $\mathcal{N}$. Thus we may assume that $(\kappa\times 1)^*\mathcal{N}$ is trivial. So $(\Lambda\times 1)^*(\kappa\times 1)^*\mathcal{N}$ is trivial too. So, since the second diagram above is commutative, $(\varphi \times 1 \times 1)^*(A_C \times 1)^*\mathcal{N}$ is trivial. Hence Equation (4.1.1) implies that $(A_C \times 1)^*\mathcal{N}$ is trivial.

Fix isomorphisms,

$$u\colon (A_C \times 1)^*\mathcal{N} \xrightarrow{\sim} \mathcal{O}_{C \times J^1_C \times T} \text{ and } v\colon (\kappa \times 1)^*\mathcal{N} \xrightarrow{\sim} \mathcal{O}_{P \times T},$$

and set $u^\dagger := (\Lambda \times 1)^* v$. Since the second diagram above is commutative, $u^\dagger$ and $(\varphi \times 1 \times 1)^* u$ differ by multiplication with an invertible (regular) function on $C^\dagger \times J^1_C \times T$. Since $C^\dagger$ is complete and integral, this function is the pullback of an invertible (regular) function on $J^1_C \times T$. Modifying u accordingly, we may assume that $u^\dagger = (\varphi \times 1 \times 1)^* u$.

Set $R := (C \times J^1_C) \times_{\bar{J}_C} (C \times J^1_C)$ and $R^\dagger := R \times_{\bar{J}_C} P$, and let $g\colon R^\dagger \to R$ be the projection. By Corollary (5.5) in [**9**], the second square in (4.1.2) is Cartesian. Hence $R^\dagger = (C^\dagger \times J^1_C) \times_P (C^\dagger \times J^1_C)$, and all three squares in the following diagram are Cartesian:

$$\begin{tikzcd}
R^\dagger \times T \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "g \times 1"] \arrow[dr, phantom, "\boxed{\square}"] & C^\dagger \times J^1_C \times T \arrow[r, "\Lambda \times 1"] \arrow[d, "\varphi \times 1 \times 1"] \arrow[dr, phantom, "\square"] & P \times T \arrow[d, "\kappa \times 1"] \\
R \times T \arrow[r, shift left] \arrow[r, shift right] & C \times J^1_C \times T \arrow[r, "A_C \times 1"] & \bar{J}_C \times T
\end{tikzcd}$$

Form the two pullbacks u_1, u_2 of u to $R \times T$ and correspondingly those $u_1^\dagger$, $u_2^\dagger$ of $u^\dagger$ to $R^\dagger \times T$. Now, $u^\dagger := (\Lambda \times 1)^* v$; so $u_1^\dagger = u_2^\dagger$.

The Abel map A_C is smooth; see (3.1). So the two projections from $R \times T$ to $C \times J^1 \times T$ are smooth too. Hence the associated points of $R \times T$ map to simple points of C. Now, φ is an isomorphism off the double point Q. Hence $g\colon R^\dagger \to R$ is an isomorphism over the associated points of R. Since

$$(g \times 1)^* u_1 = u_1^\dagger = u_2^\dagger = (g \times 1)^* u_2,$$

therefore $u_1 = u_2$ holds at every associated point of $R \times T$, so everywhere.

Since $u_1 = u_2$, by descent theory u descends to a trivialization of $\mathcal{N}$ on the image $V \times T$ of $A_C \times 1$. Now, $\bar{J}_C$ is a local complete intersection of dimension g by [**1**, (9), p. 8], where g is the arithmetic genus of C. Since each singular point of C is a double point, $\operatorname{cod}(\bar{J}_C - V, \bar{J}_C) \geq 2$ by Corollary (6.8) in [**9**]. Hence, $\mathcal{N}$ is the direct image of its restriction to $V \times T$. Therefore, $\mathcal{N}$ is trivial. The proof is now complete in the case where $\phi(T) \subset U^0$. Call this the "first case."

Using our work in the first case, we will now establish the general case. To do so, fix an arbitrary rational point $t_0 \in T$. Let $\mathcal{N}_0$ be the fiber $\mathcal{N}(t_0)$ viewed on $\bar{J}$. We will prove that $\mathcal{N}_0$ is trivial. Then we may conclude that $\phi(T) \subset U^0$, and so we will have a complete proof of the autoduality theorem of (2.1).

By hypothesis, $\mathcal{N}_0$ corresponds to a point of U. So some multiple $\mathcal{N}_0^n$ corresponds to a point of U^0. Moreover,

$$A^*_{\mathcal{L}}(\mathcal{N}_0^n) = (A^*_{\mathcal{L}} \mathcal{N}_0)^n = \mathcal{O}_C.$$

Hence, by the preceding case with S for T and $\mathcal{N}_0^n$ for $\mathcal{N}$, we may conclude that $\mathcal{N}_0^n$ is the pullback of a sheaf on S. Since S is a point, $\mathcal{N}_0^n$ is trivial on $\bar{J}$.

Consider $A^*\mathcal{N}_0$ on $C \times J^1$. It defines a map $\psi\colon J^1 \to J$ such that $\psi[\mathcal{L}] = 0$. Now, $\mathcal{N}_0^n$ is trivial. So $\psi(J^1)$ lies in the kernel of the nth power map $J \to J$. This kernel is finite. Hence $\psi(J^1) = \{0\}$ since $\psi(J^1)$ is connected and it contains 0. Hence ψ is constant since J^1 is reduced. Therefore, $A^*\mathcal{N}_0$ is equal to the pullback of some invertible sheaf $\mathcal{R}$ on J^1.

Proceed by induction on δ as in the first case, but with S for T and $\mathcal{N}_0$ for $\mathcal{N}$. If $\delta = 0$, then $U^0 = U$ as we saw above, and so $\mathcal{N}_0$ is trivial by the first case.

If $\delta \geq 1$, then the argument in the first case goes through exactly as before since the analogue of Equation (4.1.1) holds. Thus $\mathcal{N}_0$ is trivial, and the proof is now complete.

Definition (4.2). Let C/S be a flat projective family of integral curves, $\mathcal{M}$ an invertible sheaf of degree m on C/S. Define the *translation* by $\mathcal{M}$ to be the map,

$$\tau_{\mathcal{M}}\colon \bar{J}^n \to \bar{J}^{m+n},$$

given by tensoring $\mathcal{M}$ with a torsion-free sheaf.

More precisely, an S-map $t\colon T \to \bar{J}^n$ corresponds to a torsion-free rank-1 sheaf $\mathcal{N}'$ on $C \times T'/T'$ with degree-n fibers, where T'/T is an étale covering, such that the two pullbacks of $\mathcal{N}'$ to $C \times T''$ are equal, where $T''\big/ T' \times_T T'$ is an étale covering. Let $\mathcal{M}'$ be the pullback of $\mathcal{M}$ to $C \times T'$. Then $\mathcal{M}' \otimes \mathcal{N}'$ is a torsion-free rank-1 sheaf of degree $m+n$ on $C \times T'/T'$, and its two pullbacks to $C \times T''$ are equal. Hence $\mathcal{M}' \otimes \mathcal{N}'$ defines a map $\tau_{\mathcal{M}}(t)\colon T \to \bar{J}^{m+n}$.

Corollary (4.3). *Let C/S be a flat projective family of integral curves, m and n integers, and $\mathcal{M}$ an invertible sheaf of degree m on C/S. If the curves only have double points as singularities, then the translation map $\tau_{\mathcal{M}}$ induces an isomorphism,*

$$\tau_{\mathcal{M}}^*\colon \operatorname{Pic}^0_{\bar{J}^{m+n}/S} \xrightarrow{\sim} \operatorname{Pic}^0_{\bar{J}^n/S},$$

which is independent of the choice of $\mathcal{M}$. In particular, if $m = 0$, then $\tau_{\mathcal{M}}^*$ is equal to the identity on $\operatorname{Pic}^0_{\bar{J}^n/S}$.

Proof. Note that $\tau_{\mathcal{O}_C} = 1_{\bar{J}^n}$. And, if $\mathcal{M}_1$ is also an invertible sheaf on C, then

$$\tau_{\mathcal{M}} \circ \tau_{\mathcal{M}_1} = \tau_{\mathcal{M}\otimes\mathcal{M}_1}.$$

So $\tau_{\mathcal{M}}$ is an isomorphism, whose inverse is $\tau_{\mathcal{M}^{-1}}$. Hence $\tau_{\mathcal{M}}^*$ is an isomorphism. Moreover, if $\mathcal{M}_1$ is of degree m too, then $\mathcal{M} \otimes \mathcal{M}_1^{-1}$ is of degree 0, and it suffices to prove that $\tau^*_{\mathcal{M}\otimes\mathcal{M}_1^{-1}} = 1$. Thus we may assume $m = 0$.

To prove that $\tau_{\mathcal{M}}^* = 1$, we may change the base by an étale covering, and so assume that the smooth locus of C/S admits a section σ. Set $\mathcal{L} := \mathcal{O}_C(\sigma(S))$. Then $\mathcal{L}$ is an invertible sheaf on C. So,

$$\tau_{\mathcal{M}} = \tau_{\mathcal{L}^{\otimes n}} \circ \tau_{\mathcal{M}} \circ \tau_{\mathcal{L}^{\otimes -n}}.$$

Hence, since $\mathcal{L}$ is of degree 1 on C/S, we may assume $n = 0$.

Note that $\tau_{\mathcal{M}} \circ A_{\mathcal{L}} = A_{\mathcal{M}\otimes\mathcal{L}}$. Now, $A^*_{\mathcal{M}\otimes\mathcal{L}} = A^*_{\mathcal{L}}$ by Proposition (3.7). Since $A^*_{\mathcal{L}}$ is an isomorphism by the autoduality theorem, $\tau^*_{\mathcal{M}} = 1$, and the proof is complete.

Corollary (4.4). *Let C/S be a flat projective family of integral curves. If the curves only have double points as singularities, then the autoduality isomorphism $\operatorname{Pic}^0_{\bar{J}/S} \xrightarrow{\sim} J$ extends to a map $\eta\colon \overline{U} \to \bar{J}$, where $\overline{U}$ is the natural compactification of $\operatorname{Pic}^0_{\bar{J}/S}$.*

Proof. Since the curves have surficial singularities, the projective S-scheme $\bar{J}$ is flat, and its geometric fibers are integral; see [**1**, (9), p. 8]. Hence, by [**4**, Thm. (3.1), p. 28], there exists an S-scheme $U^=$ that parameterizes the torsion-free rank-1 sheaves on the fibers of $\bar{J}/S$; the connected components of $U^=$ are proper

over S. Moreover, $U^=$ contains $\mathrm{Pic}^0_{\bar{J}/S}$ as an open subscheme, and its scheme-theoretic closure in $U^=$ is, by definition, $\overline{U}$. Furthermore, since J/S is smooth and admits a section (for example, the 0-section), by [**4**, Thm. (3.4)(iii), p. 40], $\bar{J}\times\overline{U}/\overline{U}$ carries a universal sheaf $\mathcal{P}$, which is determined up to tensor product with the pullback of an invertible sheaf on $\overline{U}$.

The extension η of the autoduality map is unique, if it exists. Hence, by descent theory, it suffices to construct η after changing the base via an étale covering. So we may assume that the smooth locus of C/S admits a section σ. Set $\mathcal{L} := \mathcal{O}_C(\sigma(S))$. Then $\mathcal{L}$ is an invertible sheaf of degree 1 on C/S. So the autoduality isomorphism is simply $A^*_{\mathcal{L}}$, and it suffices to prove that $(A_{\mathcal{L}}\times 1_{\overline{U}})^*\mathcal{P}$ is a torsion-free rank-1 sheaf on $C\times\overline{U}/\overline{U}$.

The Abel map $A\colon C\times J^1\to\bar{J}$ is smooth; see (3.1). Hence $(A\times 1_{\overline{U}})^*\mathcal{P}$ is a torsion-free rank-1 sheaf on $C\times J^1\times\overline{U}/\overline{U}$. It suffices to prove that this sheaf is torsion-free rank-1 on $C\times J^1\times\overline{U}/(J^1\times\overline{U})$. The sheaf is flat over $J^1\times\overline{U}$, by the local criterion, if its fiber is flat over the fiber $J^1(u)$ for each $u\in\overline{U}$. Fix a u. Making a suitable faithfully flat base change S'/S, we may assume that the residue field of u is equal to that of its image in S. Set $\mathcal{I} := \mathcal{P}(u)$. It suffices to prove that $A(u)^*\mathcal{I}$ is a torsion-free rank-1 sheaf on $C(u)\times J^1(u)/J^1(u)$.

Suppose given an invertible sheaf $\mathcal{M}$ of degree 0 on C/S. Then the translation map $\tau_{\mathcal{M}}$ gives rise to the following commutative diagram:

$$\begin{tikzcd} C\times J^1 \arrow[r, "A"] \arrow[d, "1_C\times\tau_{\mathcal{M}}"'] & \bar{J} \arrow[d, "\tau_{\mathcal{M}}"'] \\ C\times J^1 \arrow[r, "A"] & \bar{J} \end{tikzcd}$$

By Corollary (4.3), $\tau^*_{\mathcal{M}}$ is the identity on $\mathrm{Pic}^0_{\bar{J}/S}$, so on its closure $\overline{U}$ too. Thus $\tau_{\mathcal{M}}(u)^*\mathcal{I}=\mathcal{I}$. Now, the diagram is commutative; hence,

$$(1_C\times\tau_{\mathcal{M}}(u))^*A(u)^*\mathcal{I}=A(u)^*\mathcal{I}. \tag{4.4.1}$$

Since $J^1(u)$ is integral, the lemma of general flatness applies, and it implies that there is a dense open subset W of $J^1(u)$ over which $A(u)^*\mathcal{I}$ is flat. Now, by Part (ii)(a) of Lemma (5.12) on p. 85 of [**5**], it is an open condition on the base for a flat family of sheaves to be torsion-free rank-1 provided they are supported on a family whose geometric fibers are integral of the same dimension. Hence, since $A(u)^*\mathcal{I}$ is torsion-free and of rank 1, after shrinking W, we may assume that the restriction of $A(u)^*\mathcal{I}$ to $C\times W/W$ is torsion-free rank-1. Fix a point j_1 of W and an arbitrary point j_2 of $J^1(u)$.

Making a suitable faithfully flat base change S'/S, we may assume that each of j_1 and j_2 lies in the image of a section of J^1/S. These sections represent invertible sheaves $\mathcal{M}_1$ and $\mathcal{M}_2$ of degree 1 on C/S; set $\mathcal{M} := \mathcal{M}_1\otimes\mathcal{M}_2^{-1}$. Equation (4.4.1) implies that $A(u)^*\mathcal{I}$ is torsion-free rank-1 over $\tau_{\mathcal{M}}(u)^{-1}W$ as well. Now, j_2 is an arbitrary point of $J^1(u)$. Hence $A(u)^*\mathcal{I}$ is torsion-free rank-1 on $C(u)\times J^1(u)/J^1(u)$, and the proof is complete.

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